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$\mathcal{N} = 8$ dyon partition function and walls of marginal stability

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Abstract: We construct the partition function of 1/8 BPS dyons in type II string theory on T^6 from counting of microstates of a D1-D5 system in Taub-NUT space. Our analysis extends the earlier ones by Shih, Strominger and Yin and by Pioline by taking into account the walls of marginal stability on which a 1/8 BPS dyon can decay into a pair of half-BPS dyons. Across these walls the dyon spectrum changes discontinuously, and as a result the spectrum is not manifestly invariant under S-duality transformation of the charges. However the partition function is manifestly S-duality invariant and takes the same form in all domains of the moduli space separated by walls of marginal stability, — the spectra in different domains being obtained by choosing different integration contours along which we carry out the Fourier transform of the partition function. The jump in the spectrum across a wall of marginal stability, calculated from the behaviour of the partition function at an appropriate pole, reproduces the expected wall crossing formula.

Keywords: Black Holes in String Theory, D-branes.

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1. Introduction, conventions and summary

Since the original proposal of Dijkgraaf, Verlinde and Verlinde [1] for quarter BPS dyon spectrum in heterotic string theory compactified on T^6, there has been extensive study of dyon spectrum in a variety of $\mathcal{N} = 4$ supersymmetric string theories [2 – 20]. Typically in these theories the dyon spectrum jumps discontinuously across walls of marginal stability in the moduli space on which a quarter BPS dyon can decay into a pair of half-BPS dyons. One of the important results which has emerged out of the recent studies is that the dyon partition function does not change across these walls of marginal stability. Instead in order to extract the degeneracy — or more appropriately an index — in different domains bounded by walls of marginal stability, the partition function needs to be integrated along different contours. As a result the jump in the index across a wall of marginal stability may be determined by evaluating the residue of the integrand at the pole that the contour crosses as we deform it from its initial position to the final position [13, 14, 16, 17, 19, 20].

Given this result in $\mathcal{N} = 4$ supersymmetric string theories it is natural to explore if this generalizes to other string theories with different amounts of supersymmetry. In this paper we shall extend the analysis to the 1/8 BPS dyon partition function of $\mathcal{N} = 8$ supersymmetric string theory obtained by compactifying type IIB string theory on a six dimensional torus. The dyon spectrum in this theory has been analyzed before in [21, 22, 5] using 4D-5D lift [23] and earlier results of [24] for the partition function of D1-D5 system in type IIB string theory compactified on T^5. In our analysis we include a larger set of dyons charges — including those corresponding to configurations with no D1-branes — in the definition of the partition function. This new partition function turns out to have properties similar to those of $\mathcal{N} = 4$ supersymmetric string theories, i.e. the same partition function, integrated along different contours, gives us the index of 1/8 BPS dyons in different domains in the moduli space separated by walls of marginal stability. Furthermore, the partition function has manifest S-duality invariance. We also verify explicitly that the jump in the index across a wall of marginal stability, computed from the behaviour of the partition function near an appropriate pole, reproduces the expected wall crosing formula [25 – 33].

We shall now give a brief summary of our analysis and the results. The analysis proceeds exactly parallel to that for $\mathcal{N}=4$ supersymmetric string theories described in [9–11, 18]. We consider type IIB string theory compactified on a six dimensional torus denoted as $T^4\times S^1\times \widetilde{S}^1$ and consider in this theory a 1/8 BPS configuration consisting of a Kaluza-Klein (KK) monopole — or equivalently a Taub-NUT space — associated with the $\widetilde{S}^1$ circle, a D5-brane wrapped on $T^4\times S^1$ and Q_1 D1-branes wrapped along S^1, carrying $-n$ units of momentum along S^1 and J units of momentum along $\widetilde{S}^1$. We shall call this the first description. There is a second description, obtained from the above configuration by applying an SL$(2,\mathbb{Z})$ duality of the ten dimensional type IIB string theory, followed by an $R\to 1/R$ duality on the circle $\widetilde{S}^1$ that takes it to a dual circle $\widehat{S}^1$ and finally making a six dimensional string-string duality transformation that dualizes the NS-NS 2-form field. Using the sign conventions reviewed in [18] we find that this maps the original configuration to a KK monopole associated with the circle $\widehat{S}^1$, $-Q_1$ NS 5-branes wrapped on $T^4\times S^1$, $-J$ NS 5-branes wrapped on $T^4\times\widehat{S}^1$ and a fundamental type II string wrapped on S^1, carrying $-n$ units of momentum along S^1. In the second description, we denote by Q and P the 12 dimensional electric and magnetic charge vectors in the convention where the elementary strings carry electric charges and the solitons carry magnetic charges. We also denote by L the SO$(6,6)$ continuous T-duality invariant metric of signature (6,6) and define

$$Q^2 = Q^T L Q, \qquad P^2 = P^T L P, \qquad Q\cdot P = Q^T L P\,. \tag{1.1}$$

Then for the configuration described above we have [18]

$$Q^2 = 2n, \qquad P^2 = 2Q_1, \qquad Q\cdot P = J\,. \tag{1.2}$$

In order to develop our analysis in close analogy to that in $\mathcal{N}=4$ supersymmetric string theories we shall restrict our attention to charge vectors of the type described above, or those which can be obtained from them by T-duality and electric-magnetic S-duality transformation in the second description of the theory. This allows us to have charges associated with momentum and fundamental string winding numbers along the six circles, and also Kaluza-Klein monopole and H-monopole charges associated with these circles, but does not allow us to have D-brane charges. We shall also restrict the moduli of the theory to lie in a subspace where no RR sector fields in the second description are switched on. This corresponds to requiring the states and the moduli to be invariant under $(-1)^{F_L}$ (or $(-1)^{F_R}$), and is a consistent set of restrictions. In this restricted subspace the duality group is a product of the discrete T-duality group SO$(6,6;\mathbb{Z})$ and the discrete S-duality group SL$(2,\mathbb{Z})$. Since we shall be computing an index which does not change continuously under variation of the moduli, we expect our results to be valid at least within an open neighbourhood of this restricted subspace in the full moduli space.[1]

[1] Due to the restriction on the charges our formula for the index is not manifestly invariant under the U-duality group $E_{7(7)}(\mathbb{Z})$. However we expect that it can be regarded as the special case of a more general formula which is manifestly U-duality invariant. As we shall see later, for $Q^2P^2-(Q\cdot P)^2<0$ there is an additional complication due to the existence of walls of marginal stability, — in this case the index not only depends on the charges but also on the domain of the moduli space where we compute the index.

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A 1/8 BPS state of the theory carrying the charges (Q,P) given above breaks 28 of the 32 supersymmetry generators of the theory. As a result it is accompanied by 28 real fermion zero modes and quantization of these fermion zero modes gives a 2^{14}-fold degenerate state. A generic 1/8 BPS state is obtained by taking a tensor product of this basic supermultiplet with a supersymmetry singlet state. We shall call such a state bosonic or fermionic depending on whether this supersymmetry singlet state is bosonic or fermionic, and denote by $d(Q,P)$ the number of bosonic supermultiplets minus the number of fermionic supermultiplets carrying charges (Q,P). This number can be calculated from the helicity supertrace [34, 35]

$$B_{14} = \frac{1}{14!} Tr_{Q,P}(-1)^F (2h)^{14}\,, \tag{1.3}$$

where h denotes the helicity of a state, F denotes the fermion number and $Tr_{Q,P}$ denotes trace over all states of charge (Q,P). The need for the $(2h)^{14}$ factor may be understood as follows. For every pair of fermion zero modes we have a 2-fold degeneracy with the two states carrying opposite fermion number and differing in helicity by 1/2. Thus $Tr(-1)^F$ vanishes for this pair of states and we need to insert a factor of $2h$ into the trace to get a non-zero answer. Since there are altogether 28 fermion zero modes, we need to insert 14 factors of $2h$ to 'soak up' all the fermion zero modes and give a non-vanishing answer. This explains the need for the $(2h)^{14}$ factor. Since all the factors of $2h$ are soaked up by the fermion zero modes associated with the broken supersymmetry generators, (1.3) effectively evaluates $Tr(-1)^F$ on the supersymmetry singlet states with which we tensor the basic 1/8 BPS supermultiplet. The 1/14! factor accounts for the fact that the 14 factors of $2h$ may be distributed among the traces over the 14 pairs of fermion zero modes in 14! different ways.

For the dyon configurations described earlier, $d(Q,P)$ may be regarded as a function $f(n,Q_1,J)$ of $n = Q^2/2$, $Q_1 = P^2/2$ and $J = Q\cdot P$. It now follows from the analysis of [36, 37] that the index B_{14} of any other 1/8 BPS dyon in the restricted subspace, carrying charges (Q,P), is given by $f(Q^2/2, P^2/2, Q\cdot P)$ as long as $\gcd(Q\wedge P)=1$. We define the partition function for these dyons to be

$$\begin{aligned}\Psi(\check\rho,\check\sigma,\check v) &= \sum_{n,Q_1,J} (-1)^{J+1}\, f(n,Q_1,J)\, e^{2\pi i(n\check\sigma+Q_1\check\rho+J\check v)} \\ &= \sum_{Q^2,P^2,Q\cdot P} (-1)^{Q\cdot P+1}\, f(Q^2/2,P^2/2,Q\cdot P)\; e^{i\pi(\check\sigma Q^2+\check\rho P^2+2\check v Q\cdot P)}\,.\end{aligned} \tag{1.4}$$

Conversely, we may write

$$d(Q,P) = (-1)^{Q\cdot P+1}\int_{iM_1-1/2}^{iM_1+1/2} d\check\rho \int_{iM_2-1/2}^{iM_2+1/2} d\check\sigma \int_{iM_3-1/2}^{iM_3+1/2} d\check v e^{-i\pi(\check\sigma Q^2+\check\rho P^2+2\check v Q\cdot P)}\Psi(\check\rho,\check\sigma,\check v)\,, \tag{1.5}$$

where M_1, M_2 and M_3, describing the imaginary parts of $\check\rho$, $\check\sigma$ and $\check v$, need to be fixed to values where the sum in (1.4) is convergent.

There is one subtle issue that needs to be mentioned here. The full $\mathcal{N}=8$ supersymmetry transformations acting on the fields in the restricted subspace of the second

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description, where we set the RR sector moduli and the D-brane charges to zero, will take us out of this subspace. However there are two different $\mathcal{N} = 4$ subalgebras which preserve this subspace, — one acting on the right-moving world-sheet fields and the other acting on the left-moving world-sheet fields. We shall call these right-handed and the left-handed supersymmetry algebras, — these commute with $(-1)^{F_L}$ and $(-1)^{F_R}$ symmetries respectively. A 1/8 BPS state of the full theory without any RR charge, having four unbroken supersymmetries, must correspond to a quarter BPS state of one of these two $\mathcal{N} = 4$ supersymmetry algebras. For definiteness we shall focus on those states which preserve quarter of the supersymmetries of the right-handed supersymmetry algebra. In particular in defining the partition function we shall sum over only states of this type, and not the ones which preserve quarter of the supersymmetries of the left-handed supersymmetry algebra. This corresponds to summing over states with $Q_1, n \geq 0$. We shall see however that the same partition function, expanded in a different way, also captures information about quarter BPS states of the left-handed supersymmetry algebra.

We compute the partition function Ψ by counting microstates carrying charge quantum numbers (Q, P) in the weakly coupled type II string theory in the first description. Our result for Ψ is

$$\Psi(\check{\rho}, \check{\sigma}, \check{v}) = \sum_{k\geq 0} \sum_{l\geq 0} \sum_{\substack{j\in\mathbb{Z} \\ j>0 \text{ for } k=l=0}} \left(1 - e^{2\pi i(l\check{\sigma}+k\check{\rho}+j\check{v})}\right)^{-2} e^{2\pi i(l\check{\sigma}+k\check{\rho}+j\check{v})}\, \widehat{c}(4kl - j^2)\,, \quad (1.6)$$

where $\widehat{c}(u)$ is defined through the relations [24, 21]

$$-\vartheta_1(z|\tau)^2\, \eta(\tau)^{-6} \equiv \sum_{k,l} \widehat{c}(4k - l^2)\, e^{2\pi i(k\tau + lz)}\,. \quad (1.7)$$

$\vartheta_1(z|\tau)$ and $\eta(\tau)$ are respectively the odd Jacobi theta function and the Dedekind eta function. The $k \geq 1$ terms in the sum in (1.6) are identical to the partition function of the D1-D5 system in type IIB string theory compactified on $T^4 \times S^1$ [24]. As we shall explain, the $k = 0$ term comes from the dynamics of a single D5-brane moving in the KK monopole background and is essential for S-duality invariance of the partition function and consistency with the wall crossing formula. In this context we also note that in the $k = l = 0$ term if we had put the restriction $j < 0$ instead of $j > 0$ we would have gotten the same analytic function.

The partition function (1.6) itself does not give the index $d(Q, P)$ unambiguously; we must specify the values of M_1, M_2 and M_3 appearing in (1.5) or equivalently describe how we should Fourier expand Ψ to extract the index. It turns out that in the weakly coupled type IIB string theory in the first description where we have carried out the computation, we need to first expand Ψ in powers of $e^{2\pi i\check{\rho}}$ and $e^{2\pi i\check{\sigma}}$ and then expand the coefficient of each term in powers of $e^{2\pi i\check{v}}$ or $e^{-2\pi i\check{v}}$. This unambiguously determines the contribution from the $(k, l) \neq (0, 0)$ terms, — for any given power of $e^{2\pi i\check{\rho}}$ and $e^{2\pi i\check{\sigma}}$ the power of $e^{2\pi i\check{v}}$ is bounded both from above and below and the Fourier expansion in $e^{2\pi i\check{v}}$ is unambiguous. However for the $k = l = 0$ term the summand can be expanded either as a power series expansion in $e^{2\pi i\check{v}}$ or as a power series expansion in $e^{-2\pi i\check{v}}$, with infinite number of terms in

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each case. The correct choice depends on whether the angle θ between S^1 and $\widetilde{S}^1$ is larger than 90° or less than 90°. In other words, the actual value of the index changes as we cross the $\theta = 90°$ line. This is entirely analogous to the corresponding result found in [9] and reviewed in detail in [18] for $\mathcal{N} = 4$ supersymmeric string theories. Equivalently, while determining $d(Q, P)$ via eq. (1.5), the quantities M_1, M_2 and M_3 which fix the integration contour need to be chosen in the range

$$M_1, M_2 \gg |M_3| \gg 0\,, \tag{1.8}$$

with the sign of M_3 determined by the angle between S^1 and $\widetilde{S}^1$. If we try to deform the contour for $M_3 > 0$ to the one for $M_3 < 0$ we pick up the residue at the pole of $\Psi(\check{\rho}, \check{\sigma}, \check{v})$ at $\check{v} = 0$, and as a result the index changes by an amount determined from the residue at the pole. This correctly accounts for the change in the index across the $\theta = 90°$ line.

Our analysis determines the form of the partition function in a specific domain — more precisely two domains separated by the $\theta = 90°$ line — of the moduli space corresponding to weak string coupling in the first description of the theory. *A priori* the index $d(Q, P)$ and the partition function Ψ could have different forms in other domains in the moduli space separated by walls of marginal stability. However as in the case of all known examples in $\mathcal{N} = 4$ supersymmetric string theories, we find that the partition function of the $N = 8$ supersymmetric string theory is also described by the same analytic function in all domains of the moduli space. The choice of the integration contour, encoded in the choice of M_1, M_2 and M_3, differs in different domains separated by the walls of marginal stability. The proof of this follows the same logic as in the case of $\mathcal{N} = 4$ supersymmetric string theories [13, 18]. First of all we note that S-duality transformations take us from one domain bounded by walls of marginal stability to another such domain [13, 14]. In particular if the original domain is bounded by walls of marginal stability on which the dyon of charge (Q, P) decays into dyons of charge $(\alpha_i Q + \beta_i P, \gamma_i Q + \delta_i P)$ and $((1 - \alpha_i)Q - \beta_i P, -\gamma_i Q + (1 - \delta_i)P)$, then S-duality transformation of the charges and the moduli by the SL$(2, \mathbb{Z})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps it into a new domain in which $\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$ is replaced by [13, 37]

$$\begin{pmatrix} \alpha_i' & \beta_i' \\ \gamma_i' & \delta_i' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} . \tag{1.9}$$

Thus we can find the index for the dyons in other domains by an S-duality transformation of our original formula. On the other hand explicit S-duality transformation of (1.5) by the SL$(2, \mathbb{Z})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ leads to a new formula for the index where we replace in (1.5) $\Psi(\check{\rho}, \check{\sigma}, \check{v})$ and (M_1, M_2, M_3) by [13, 19]

$$\Psi(d^2\check{\rho} + b^2\check{\sigma} + 2bd\check{v}, c^2\check{\rho} + a^2\check{\sigma} + 2ac\check{v}, cd\check{\rho} + ab\check{\sigma} + (ad + bc)\check{v}) \tag{1.10}$$

and

$$(a^2M_1 + b^2M_2 - 2abM_3, c^2M_1 + d^2M_2 - 2cdM_3, -acM_1 - bdM_2 + (ad + bc)M_3) \tag{1.11}$$

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respectively. Thus if we can show that the partition function Ψ is manifestly invariant under the S-duality transformation:

$$\begin{aligned}\Psi(\check\rho,\check\sigma,\check v) &= \Psi(d^2\check\rho+b^2\check\sigma+2bd\check v, c^2\check\rho+a^2\check\sigma+2ac\check v, cd\check\rho+ab\check\sigma+(ad+bc)\check v),\\ &\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z})\,,\end{aligned}\tag{1.12}$$

then the result for the index in different domains in the moduli space will be given just by changing the integration contour according to (1.11). We prove (1.12) explicitly in § 3.

In order to use (1.11) to find the integration contour in different domains in the moduli space we need information about the choice of contour in at least one such domain. This can be gotten from the results of weak coupling calculation in the first description. In particular if we consider the domain bounded by the walls corresponding to the decays $(Q,P)\to(Q,0)+(0,P)$, $(Q,P)\to(Q,Q)+(0,P-Q)$ and $(Q,P)\to(Q-P,0)+(P,P)$, then we need to choose the M_i's as $M_1, M_2 \gg -M_3 \gg 0$ [13, 18].[2] An explicit prescription for (M_1,M_2,M_3) at different points in the moduli space satisfying (1.11) can be found in [17, 19].

The S-duality invariance of Ψ allows us to calculate the jump in the index across a wall of marginal stability by calculating the residue at the pole(s) the contour crosses as we deform it from the initial position to the final position calculated according to (1.11). It follows from the corresponding analysis in $\mathcal{N}=4$ supersymmetric string theories [13, 19, 20] that as we cross a wall of marginal stability associated with the decay into a pair of half-BPS states[3]

$$(Q,P) \to (Q_1,P_1)+(Q_2,P_2)\,,\tag{1.13}$$

$$(Q_1,P_1) = (\alpha Q+\beta P,\gamma Q+\delta P),\qquad (Q_2,P_2)=(\delta Q-\beta P,-\gamma Q+\alpha P)\,,\tag{1.14}$$

$$\alpha\delta = \beta\gamma,\qquad \alpha+\delta=1,\qquad \alpha,\beta,\gamma,\delta\in\mathbb{Z}\,,\tag{1.15}$$

the contour crosses the pole at

$$\check\rho\gamma-\check\sigma\beta+\check v(\alpha-\delta)=0\,.\tag{1.16}$$

[2] Although we are using these decays to label different domains in the moduli space, we shall see in § 4 that the actual jump in the index across many of these walls vanishes in the $\mathcal{N}=8$ supersymmetric string theory.

[3] It follows from the analysis of [13, 38 − 40] that in the restricted subspace of the moduli space we are considering, such decays occur on codimension 1 subspaces. This is a necessary (but not sufficient) condition for an index to jump across this subspace, since for higher codimension subspaces we can avoid the change in the index by *going around the subspace*. We shall see later that for a subset of the decays of the type described in (1.13)–(1.15) for which $(\alpha Q+\beta P)$ and $(\delta Q-\beta P)$ are null and as a result the final decay products are half-BPS under the full $\mathcal{N}=8$ supersymmetry algebra and not just the $\mathcal{N}=4$ subalgebra that we are considering here, the index B_{14} jumps discontinuously. Thus we expect that such decays will continue to occur on a codimension one subspace of the full moduli space even after turning on the RR sector moduli. For the other decays of the type described in (1.13)–(1.15) for which either $(\alpha Q+\beta P)$ or $(\delta Q-\beta P)$ fails to be null, the change in B_{14} vanishes. Thus the associated subspace may or may not remain codimension one in the full moduli space. Finally, if we consider decays which fail to satisfy (1.14) and/or (1.15), then it follows from the analysis of [13, 38 − 40] that such decays occur on subspaces of codimension >1 in the restricted subspace of the moduli space we are considering and hence also in the full moduli space. The change in B_{14} associated with such decays always vanishes.

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Thus the jump in the index across a wall of marginal stability is given by the residue of the integrand in (1.5) at the pole at (1.16). We check in § 4 that the jump in the index predicted by this formula at the wall associated with the decay $(Q,P) \to (Q,0) + (0,P)$ reproduces the expected change in the index across a wall of marginal stability [25 – 33]. Similar agreement at the other walls of marginal stability then follows from S-duality invariance of Ψ.

Refs. [21, 22] proposed formulæ similar to the one given in (1.6) for the 1/8 BPS dyon spectrum in type IIB string theory on T^6. However both these proposals treated the dyon spectrum as universal independent of the region of the moduli space where we calculate the spectrum. Thus in any given region in the moduli space these proposals differ from ours in a subtle way. As we shall see below eq. (4.1), the index jumps discontinuously across a wall of marginal stability only for states with $Q^2P^2 - (Q \cdot P)^2 < 0$. Thus the subtle difference between our results and those proposed in [21, 22] arises only for such states.

Since for positive $Q^2P^2 - (Q \cdot P)^2$ we do not have any associated wall of marginal stability, the index calculated from (1.5), (1.6) is independent of the choice of (M_1, M_2, M_3) and the formula for the index $d(Q,P)$ takes the form [24, 22]

$$d(Q,P) = \sum_{s|Q^2/2, P^2/2, Q\cdot P; s>0} s\, \widehat{c}\left(\frac{Q^2P^2 - (Q\cdot P)^2}{s^2}\right) . \tag{1.17}$$

This formula is manifestly S-duality invariant since $Q^2P^2 - (Q\cdot P)^2$ and $\gcd(Q^2/2, P^2/2, Q\cdot P)$ are S-duality invariants.

The partition function (1.6) has been derived for the 1/8 BPS states whose unbroken supersymmetries lie in the right-handed $\mathcal{N} = 4$ supersymmetry algebra. The index for 1/8 BPS states whose unbroken supersymmetries lie in the left-handed $\mathcal{N} = 4$ supersymmetry algebra can be obtained from the former by world-sheet parity transformation exchanging these two supersymmetry algebras. This effectively replaces the $O(6,6)$ invariant metric L by $-L$ in all formulæ. In particular this changes $(Q^2, P^2, Q\cdot P)$ to $(-Q^2, -P^2, -Q\cdot P)$. Thus the new partition function is related to the previous one by the transformation $(\check\rho, \check\sigma, \check v) \to (-\check\rho, -\check\sigma, -\check v)$. This however produces the same analytic function $\Psi(\check\rho, \check\sigma, \check v)$ since each term in the sum in (1.6) is invariant under the transformation $(\check\rho, \check\sigma, \check v) \to (-\check\rho, -\check\sigma, -\check v)$. Thus the same partition function $\Psi(\check\rho, \check\sigma, \check v)$ contains information about both types of 1/8 BPS states, but the choice of (M_1, M_2, M_3) in (1.5) for these two types of dyons are related by the transformation $(M_1, M_2, M_3) \to (-M_1, -M_2, -M_3)$, accompnied by $L \to -L$ replacement in the expression for M_i's in terms of the moduli given in [17, 19]. Also for a given charge vector (Q,P), the physical location of the walls of marginal stability in the moduli space, computed in [13], will differ for the two algebras due to the $L \to -L$ replacement in the associated formulæ.

Finally we note that some of the $\mathcal{N} = 2$ supersymmetric models discussed in [41] have properties close to that of $\mathcal{N} = 4$ supersymmetric string theories. It will be interesting to explore if the phenomenon of having a universal partition function independent of the domains in the moduli space holds for these theories as well.

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2. Microstate counting

In this section we shall describe how we arrive at the formula (1.6) for the dyon partition function. We shall carry out our analysis in the weak coupling limit of the first description of the theory, regarding it as a collection of KK monopole – D1-D5 system carrying momenta along the circles S^1 and $\widetilde{S}^1$. As in [9], in this limit we can regard the degrees of freedom of the KK monopole, the D1-D5 center of mass motion and the D1-D5 relative motion as three non-interacting systems and calculate the partition function of the combined system by taking the product of the three partition functions.

Since we are computing the partition function of 1/8 BPS states, special care must be taken to ensure that all the fermion zero modes associated with the 28 broken supersymmetry generators are soaked up. As described in § 1, each pair of fermion zero modes are soaked up by an insertion of $(-1)^F 2h$ factor in the helicity trace. Let us examine how the fermion zero modes get distributed between the three subsystems described above. First of all the KK monopole associated with $\widetilde{S}^1$ breaks 16 of the 32 supersymmetries of the original theory. Thus the world-volume theory on the KK monopole has 16 fermion zero modes which must be soaked up by a factor of $(-1)^F(2h)^8$ inserted into the partition function. The classical D5-brane in the KK monopole background breaks 8 of the remaining 16 supersymmetries and hence its world-volume carries 8 fermion zero modes. We need a factor of $(-1)^F(2h)^4$ inserted into the partition function of the D5-brane to soak up these zero modes. Finally it is known from the analysis of [24] that the system describing the dynamics of the D1-branes inside the D5-brane has four fermion zero modes and hence requires a factor of $(-1)^F(2h)^2$ inserted into the partition function to soak up these zero modes. This determines how the 14 factors of $(2h)$ are distributed among different components of the partition function.

The above analysis also shows that in order to get a non-vanishing result for the index we must keep both the KK monopole and the D5-brane in their ground state. In particular we cannot excite any mode carrying momentum along S^1. If we did, then such a system will break more supersymmetries and we shall need additional factors of $(2h)$ to be inserted into the partition function in order to get a non-vanishing result. Since the total power of $2h$ is fixed to be 14, this would require removing some factors of $2h$ from some other component of the partition function, making the corresponding contribution vanish.[4] Thus the non-trivial contribution to the partition function comes solely from the relative motion of the D1-D5 system. This was evaluated in [24], yielding the answer[5]

$$\sum_{k\geq 1}\sum_{l\geq 0}\sum_{j\in\mathbb{Z}}\left(1-e^{2\pi i(l\check{\sigma}+k\check{\rho}+j\check{v})}\right)^{-2} e^{2\pi i(l\check{\sigma}+k\check{\rho}+j\check{v})}\,\widehat{c}(4kl-j^2)\,, \tag{2.1}$$

where $\widehat{c}(u)$ has been defined in eq. (1.7). This can be identified with the contribution to (1.6) from the $k\geq 1$ terms.

[4] This should be contrasted with the corresponding analysis in $\mathcal{N}=4$ supersymmetric string theories where the KK monopole and the D5-brane can have excitations carrying left-moving momentum along S^1 without breaking any additional supersymmetry.

[5] The $(-1)^{Q\cdot P}$ factor in eqs. (1.4), (1.5) relating the index to the partition function has the same origin as in the case of $\mathcal{N}=4$ supersymmetric string theories [5, 17, 18] and will not be discussed here.

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In arriving at (2.1) we have implicitly assumed that leaving aside the degeneracy implied by the 8 broken supersymmetry generators, there is a unique supersymmetric state describing the D5-brane bound to the KK monopole. Is this true? A D5-brane in flat space has altogether 16 fermion fields on its world-volume. As pointed out before, classically a D5-brane in the background of the KK monopole has eight unbroken supersymmetries and breaks eight supersymmetries. Thus the world-volume theory of the D5-brane should contain eight free goldstino fermions. These, together with the four scalar fields associated with the Wilson lines along T^4 describe a free (4,4) supersymmetric field theory in 1+1 dimensions spanning the time coordinate and the coordinate along S^1. The rest of the eight fermion fields on the D5-brane world-volume combine with the four scalar fields describing motion along the Taub-NUT space to give an interacting (4,4) supersymmetric sigma model with Taub-NUT target space. By the standard argument the number of supersymmetric ground states of this system is equal to the number of harmonic forms on the Taub-NUT space which is 1 [42, 43]. Since the harmonic form is normalizable the state describes a bound state [44, 45]. This confirms our implicit assumption that leaving aside the degeneracy implied by the 8 broken supersymmetry generators, there is a unique supersymmetric state describing the D5-brane bound to the KK monopole.

The above analysis holds when the S^1 and $\widetilde{S}^1$ circles are orthogonal. When we switch on a modulus that changes the angle between S^1 and $\widetilde{S}^1$ there is an attractive force between the D5-brane and the KK monopole, giving rise to additional potential terms [9]. Since in the limit in which the size of the Taub-NUT space is large this potential term as well as the Taub-NUT metric affects mainly the modes independent of the S^1 coordinate, we can treat the non-zero mode oscillators carrying momentum along S^1 as free and focus our attention on the supersymmetric quantum mechanics describing the zero modes. The latter is obtained via dimensional reduction of the 1+1 dimensional field theory to 0+1 dimensions. This system was analyzed in [46 − 48] in the context of quarter BPS dyon spectrum in the $N = 4$ supersymmetric gauge theories and the result may be summarized as follows. Besides the fully supersymmetric ground state described above the system has a set of states where 4 of the eight remaining supersymmetries are broken. Depending on the angle between S^1 and $\widetilde{S}^1$ these partially supersymmetric states have either only positive or only negative momentum along $\widetilde{S}^1$, and there are precisely $|j|$ states carrying $\widetilde{S}^1$ momentum j.[6] Thus the partition function for these dyons can be written as

$$\sum_{j=1}^{\infty} j e^{\pm 2\pi i \check{v} j} = e^{2\pi i v}/(1 - e^{2\pi i v})^2 \,. \tag{2.2}$$

Note that the result for the partition function is independent of whether we use positive or negative momentum along $\widetilde{S}^1$.

The partially supersymmetric states described above are not relevant for the computation at hand since due to the four additional broken supersymmetries we need to insert a

[6]We are counting only those states which, from the space-time viewpoint, preserve quarter of the supersymmetries of the right-handed $\mathcal{N} = 4$ supersymmetry algebra. There are also states preserving quarter of the supersymmetries of the left-handed supersymmetry algebra; for these states the sign of $\widetilde{S}^1$ momentum is opposite.

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factor of $(-1)^F(2h)^2$ into the partition function. This effectively uses up all the factors of $(2h)$ in order to saturate the fermion zero modes of the KK monopole - D5-brane system, and there is no left over factors of $2h$ which we can insert into the partition function of the D1-D5 system. As a result the contribution from such terms vanish.

We must note however that in computing the full partition function we must also include states with $Q_1 = 0$, i.e. no D1-branes if there are 1/8 BPS states in this sector. Proceeding as before we can see that the KK monopole soaks up a factor of $(2h)^8$ and the 8 broken supersymmetries associated with the classical D5-brane soak up another factor of $(2h)^4$. Thus we are left with a factor of $(2h)^2$. However unlike in the $Q_1 \neq 0$ case where these factors needed to be inserted in the D1-D5 partition function to get a non-vanishing contribution, here we are free to use them to either get additional excitations on the D5-brane from the left-moving modes[7] of the D5-brane world-volume theory carrying negative momenta along S^1, keeping the zero-mode part of the theory in the fully supersymmetric ground state, or keeping the non-zero modes in their ground state and using one of the partially supersymmetric bound states of the zero-mode part of the D5-brane world-volume theory. The contribution from the first set of states can be computed by recalling from [24] that the left-moving fields on the D5-brane world-volume have the following $(j, 2h)$ assignment: four scalars with $(j, 2h) = (0, 0)$ representing the Wilson lines along T^4, one scalar each for $(j, 2h) = (1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$ representing the coordinates along the Taub-NUT space, and 2 fermions each for $(j, 2h) = (1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$. Following [24] this yields the answer

$$\begin{aligned}\frac{1}{2}\frac{d^2}{d\widetilde{y}^2}\prod_{l=1}^{\infty}\Bigg[(1-q^l)^{-4}\Big\{\prod_{j=\pm 1}\prod_{\widetilde{j}=\pm 1}(1-q^l y^j \widetilde{y}^{\widetilde{j}})^{-1}\Big\}\times \\ \times\Big\{\prod_{j=\pm 1}(1-q^l y^j)^2\Big\}\Big\{\prod_{\widetilde{j}=\pm 1}(1-q^l \widetilde{y}^{\widetilde{j}})^2\Big\}\Bigg]\Bigg|_{\widetilde{y}=1}, \quad q \equiv e^{2\pi i\check{\sigma}}, \quad y \equiv e^{2\pi i\check{v}}\,. \end{aligned} \tag{2.3}$$

Explicit evaluation of this term leads to

$$\sum_{l=1}^{\infty}\left[-2\left(1-e^{2\pi i l\check{\sigma}}\right)^{-2} e^{2\pi i l\check{\sigma}} + \sum_{j=\pm 1}\left(1-e^{2\pi i(l\check{\sigma}+j\check{v})}\right)^{-2} e^{2\pi i(l\check{\sigma}+j\check{v})}\right]. \tag{2.4}$$

Using the fact that $\widehat{c}(u) = 0$ for $u < -1$ and $\widehat{c}(0) = -2$, $\widehat{c}(-1) = 1$, (2.4) can be identified with the $k = 0$, $l \geq 1$ terms in the sum in (1.6).

Finally we turn to the contribution to the partition function from the partially supersymmetric bound states of the D5-brane to the KK monopole. To evaluate this contribution we note that since the extra factor of $(2h)^2$ is now absorbed by the supersymmetric quantum mechanics describing the D5-brane motion in the KK monopole background, we need to evaluate the trace over the non-zero mode oscillators without any factor of $2h$. The corresponding contribution is given by (2.3) without the $\frac{1}{2}\partial_{\widetilde{y}}^2$ operation. At $\widetilde{y} = 1$ this

[7] Exciting the right-moving modes will break the right-handed $\mathcal{N} = 4$ supersymmetry algebra but preserve the left-handed $\mathcal{N} = 4$ supersymmetry algebra. As mentioned in § 1, we shall not include such states in the definition of the partition function even though they describe 1/8 BPS dyons.

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reduces to 1, indicating that we cannot excite any of the oscillators carrying momentum along S^1. Thus the contribution to the partition function from this set of states is given by (2.2). This can be identified as the $k = 0$, $l = 0$ term in (1.6).

The sum of the expressions given in (2.1), (2.4) and (2.2) gives the complete contribution to the partition function stated in (1.6).

3. S-duality invariance of the partition function

In this section we shall give a proof of S-duality invariance of the partition function encoded in (1.12). This requirement may be expressed as

$$\Psi(\check\rho, \check\sigma, \check v) = \Psi(\check\rho', \check\sigma', \check v')\,, \tag{3.1}$$

where

$$\begin{aligned} \check\rho' &= d^2\check\rho + b^2\check\sigma + 2bd\check v, \\ \check\sigma' &= c^2\check\rho + a^2\check\sigma + 2ac\check v, \\ \check v' &= cd\check\rho + ab\check\sigma + (ad+bc)\check v, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,. \end{aligned} \tag{3.2}$$

The proof of (3.1) goes as follows. We first note that

$$k\check\rho + l\check\sigma + j\check v = k'\check\rho' + l'\check\sigma' + j'\check v'\,, \tag{3.3}$$

where

$$k' = a^2k + c^2l - acj, \quad l' = b^2k + d^2l - bdj, \quad j' = -2abk - 2cdl + (ad+bc)j\,. \tag{3.4}$$

Furthermore we have

$$4k'l' - j'^2 = 4kl - j^2\,. \tag{3.5}$$

Using these relations in (1.6) we get

$$\Psi(\check\rho, \check\sigma, \check v) = \sum_{k\geq 0}\sum_{l\geq 0}\sum_{\substack{j\in\mathbb{Z} \\ j>0 \text{ for } k=l=0}} \left(1 - e^{2\pi i(l'\check\sigma' + k'\check\rho' + j'\check v')}\right)^{-2} e^{2\pi i(l'\check\sigma' + k'\check\rho' + j'\check v')}\,\widehat{c}(4k'l' - j'^2)\,. \tag{3.6}$$

If we can now show that k', l' and j' take values over the same range as k, l and j, then the right hand side of (3.6) can be identified as $\Psi(\check\rho', \check\sigma', \check v')$. This would prove the desired relation (3.1). We shall prove this separately for the S-duality transformations $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since these two matrices generate the whole $\mathrm{SL}(2,\mathbb{Z})$ group, once we have proven invariance of Ψ under S and T, it proves invariance of Ψ under the full $\mathrm{SL}(2,\mathbb{Z})$ S-duality group.

First consider the transformation by S. In this case (3.4) takes the form

$$k' = l, \quad l' = k, \quad j' = -j\,. \tag{3.7}$$

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Thus k' and l' run over non-negative integers. Furthermore for $(k', l') \neq (0,0)$, j' can take arbitrary integer values, whereas for $k' = l' = 0$, j' takes only negative integer values. Using $j' \to -j'$ symmetry of the summand in (3.6) for $k' = l' = 0$ we can turn this into sum over positive integer values of j'. Thus the range of (k', l', j') coincides with the range of summation over (k, l, j). This proves invariance of Ψ under the transformation S.

We now turn to the transformation T. In this case (3.4) gives

$$k' = k, \quad l' = k + l - j, \quad j' = j - 2k\,. \tag{3.8}$$

Using the $j \to -j$ invariance of the $k = l = 0$ term in (1.6) we shall take the sum over the original variable j to run over negative integer values. We now see from (3.8) that k' always takes non-negative integer values. This agrees with the corresponding range of values of k. To determine the range of values taken by l' we shall consider two cases separately. First consider the case $k = l = 0$. In this case $l' = -j$. Since $j < 0$, we see that l' takes positive integer values. On the other hand for $(k, l) \neq (0, 0)$ we have

$$(k + l)^2 - j^2 = (k - l)^2 + 4kl - j^2 \geq -1\,, \tag{3.9}$$

where in the last step we have used the fact that $j^2 \leq 4kl + 1$ for $\widehat{c}(4kl - j^2)$ to be non-zero. Furthermore the inequality (3.9) is saturated only if $k = l$ and $j^2 = 4kl + 1 = 4k^2 + 1$. Since $j, k, l \in \mathbb{Z}$, the only solution to this is $k = l = 0$, $j = \pm 1$. This contradicts our assumption that $(k, l) \neq (0, 0)$. Thus we see that the bound cannot be saturated and we must have

$$(k + l)^2 \geq j^2\,. \tag{3.10}$$

Since k, l are non-negative integers this gives $k + l \geq |j|$. Thus l' given in (3.8) is a non-negative integer.

Next we shall examine the range of j' for $k' = l' = 0$. From (3.8) we see that in this case $k = 0$, $j = l$ and $j' = j = l$. Furthermore l cannot vanish, since for $k = l = 0$, j must be a negative integer and hence we cannot have $j = l$. This shows that l and hence $j' = l$ must be a positive integer. Thus the allowed range of values of k', l' and j' takes the form

$$k' \geq 0, \quad l' \geq 0, \quad j' \in \mathbb{Z}, \quad j' > 0 \text{ for } k' = l' = 0\,. \tag{3.11}$$

This coincides with the allowed range of values of k, l and j after flipping the sign of j for $k = l = 0$. As argued before, the latter operation preserves the form of Ψ.

In order to complete the proof of invariance of Ψ under the transformation T we must now verify that all the values in the range (3.11) are realized, i.e. given any triplet (k', l', j') subject to the condition (3.11) we can find a triplet (k, l, j) satisfying (3.8) within the allowed range. For this we need to invert (3.8) as

$$k = k', \quad l = k' + l' + j', \quad j = j' + 2k'\,, \tag{3.12}$$

and repeat our arguments in the opposite direction. This is a straightforward exercise and we find that for any triplet k', l' and j' satisfying (3.11), k and l given in (3.12) are non-negative integers, j is an integer, and for $k = l = 0$, j is a positive integer which

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can be turned into a negative integer using the $j \to -j$ symmetry of the summand. This completes the proof of invariance of Ψ under the S-duality transformation T.

This establishes S-duality invariance of Ψ given in (1.12). Note that inclusion of the $k = 0$ terms in (1.6), which count the contribution from 1/8 BPS dyons with vanishing D1-brane charge, is essential for getting an S-duality invariant partition function. Without this term the range of k', l' and j' will not coincide with that of k, l and j.

4. Wall crossing

In this section we shall carry out an independent computation of the change in the index of a 1/8 BPS dyon as we cross a wall of marginal stability at which the 1/8 BPS dyon breaks up into a pair of half-BPS dyons, and compare the result with the result predicted from (1.6).

Let us first compute the jump in the index predicted by (1.6). According to eqs. (1.13)–(1.16), as we cross the wall of marginal stability associated with the decay $(Q,P) \to (Q,0) + (0,P)$, the contour will cross the pole at $\check{v} = 0$. Such poles of Ψ come from the $k = l = 0$ term in (1.6). Thus the change in the index as we cross this wall is given by the residue of the integrand in (1.5) at the pole at $\check{v} = 0$. This gives

$$\Delta d(Q,P) = (-1)^{Q\cdot P+1}\, Q\cdot P\, \widehat{c}(-1)\, \delta_{Q^2,0}\, \delta_{P^2,0} = (-1)^{Q\cdot P+1}\, Q\cdot P\, \delta_{Q^2,0}\, \delta_{P^2,0}\,. \tag{4.1}$$

Using S-duality invariance we see that for a more general decay of the type given in (1.13)–(1.15) we must have $(\alpha Q + \beta P)^2 = 0$ and $(\delta Q - \beta P)^2 = 0$ for $\Delta d(Q,P)$ to be non-zero. Combining these with the constraints (1.15) we get

$$\begin{aligned} \alpha &= \frac{1}{2} - \frac{Q\cdot P}{2\sqrt{(Q\cdot P)^2 - Q^2P^2}}, \qquad \beta = \frac{Q^2}{2\sqrt{(Q\cdot P)^2 - Q^2P^2}}, \\ \gamma &= -\frac{P^2}{2\sqrt{(Q\cdot P)^2 - Q^2P^2}}, \qquad \delta = \frac{1}{2} + \frac{Q\cdot P}{2\sqrt{(Q\cdot P)^2 - Q^2P^2}}\,. \end{aligned} \tag{4.2}$$

Since $\alpha, \beta, \gamma, \delta$ must be integers, this gives strong constraints on Q^2, P^2 and $(Q\cdot P)^2$. In particular $Q^2P^2 - (Q\cdot P)^2$, known as the discriminant, must be negative. Furthermore for a given (Q,P) there is at most a single wall of marginal stability corresponding to α, β, γ, δ given by (4.2). This is an enormous simplification compared to the corresponding results in $\mathcal{N} = 4$ supersymmetric string theories [13].

Note that (1.6) has a pole at $k\check{\rho} + l\check{\sigma} + j\check{v} = 0$ for every (k,l,j) in the range of summation given in (1.6). Comparing this to (1.16) we see that the poles given in (1.16) arise from the $(k,l,j) = (\gamma, -\beta, \alpha - \delta)$ term in the sum in (1.6). The restrictions (1.15) on $\alpha, \beta, \gamma, \delta$ then pick out those (k,l,j) for which $4kl - j^2 = -1$. Can the poles associated with other values of (k,l,j) cause further jumps in the index? To answer this question we note that given that the original contour, appropriate to the weak coupling region of the first description, has $M_1, M_2 \gg |M_3| > 0$, it satisfies

$$M_1, M_2 > 0, \qquad M_1M_2 > M_3^2\,. \tag{4.3}$$

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Since the conditions (4.3) are invariant under the S-duality transformation (1.11), all the contours corresponding to different domains of the moduli space satisfy (4.3). It is now easy to see that while deforming a contour associated with one such (M_1, M_2, M_3) to a contour associated with another (M_1, M_2, M_3) satisfying (4.3), we never hit the pole at $k\check{\rho} + l\check{\sigma} + j\check{v} = 0$ unless $4kl - j^2 < 0$, — indeed for $4kl - j^2 \geq 0$, $\Im(k\check{\rho} + l\check{\sigma} + j\check{v})$ is positive on the initial and the final contours and hence can be made to remain positive during the deformation. Since $\widehat{c}(u) = 0$ for $u < -1$, this gives $4kl - j^2 = -1$ as the condition for hitting the pole. All such poles are of the form given in (1.15).

For an independent computation of the jump in the index across the wall of marginal stability associated with the decay $(Q, P) \to (Q, 0) + (0, P)$, we use the following argument [25−31, 33]. Let us recall first the corresponding computation in $\mathcal{N} = 2$ supersymmetric string theories where a half BPS state breaks 4 of the 8 supersymmetries and hence has 4 fermion zero modes. Now consider a wall of marginal stability where a half BPS state decays into a pair of half BPS states. On one side of the wall the decay products form a bound state with large separation between the components. This bound state ceases to exist on the other side of the wall causing a jump in the index. Thus the change in the index can be computed by computing the index associated with such bound states. Although naively the system has 8 fermion zero modes — 4 associated with each component — four of these zero modes take part in the interaction which form the bound state. Thus we are left with four fermion zero modes as is expected of a single half-BPS state of the theory. The detailed analysis of the quantum mechanics leading to the formation of the bound state yields a multiplicative factor of $(-1)^{Q\cdot P+1}\, Q \cdot P$ to the index of these bound states, thereby leading to the formula

$$\Delta d(Q, P) = (-1)^{Q\cdot P+1}\, Q \cdot P\, d_h(Q, 0)\, d_h(0, P)\,, \tag{4.4}$$

for the jump in the index. Here $d_h(Q, 0)$ and $d_h(0, P)$ denote the index of half-BPS states carrying charges $(Q, 0)$ and $(0, P)$ respectively and represent the contribution to the bound state index due to the internal degeneracy of each component.

If we consider the decay of a quarter BPS dyon of $\mathcal{N} = 4$ supersymmetric string theory into a pair of half-BPS dyons, then the following argument due to Denef [33] may be used to generalize the above result. A quarter BPS state in $\mathcal{N} = 4$ supersymmetric string theory breaks 12 of the 16 supersymmetries and hence has 12 fermion zero modes. On the other hand the pair of half-BPS states each carry 8 fermion zero modes. Of the 16 fermion zero modes associated with the pair, 4 are used up in formation of the bound state as before, leaving us with 12 fermion zero modes required to give rise to a quarter BPS supermultiplet. As before supersymmetric quantum mechanics produces a degeneracy factor of $(-1)^{Q\cdot P+1}\, Q \cdot P$ and leads to (4.4) for the change in the index across a wall of marginal stability. Note that if we had replaced one or both of the decay products by a quarter BPS state then the resulting system would have too many fermion zero modes and hence the index would vanish [33]. This shows that in $\mathcal{N} = 4$ supersymmetric string theories the change in the index occurs only across walls where a quarter BPS dyon decays into a pair of half BPS dyons.

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Let us now apply the same line of argument to the decay of a 1/8 BPS state in $\mathcal{N}=8$ supersymmetric string theories. As pointed out before, such a dyon is accompanied by 28 fermion zero modes. On the other hand a pair of half-BPS states in this theory carries a total of 32 fermion zero modes of which 4 are used up for bound state formation. Thus we are again left with the correct number of fermion zero modes and the quantum mechanics of bound states leads to the degeneracy factor of (4.4). We now note that for the class of charge vectors we are considering, — which in the second description involve charges carried by elementary string states and solitons without any D-branes charges, — the only half BPS states are those associated with elementary string states carrying charge $(Q,0)$ with $Q^2=0$ (i.e. no left- or right-moving world-sheet excitation) and their S-duals carrying charges (aQ,cQ) with $Q^2=0$, $\gcd(a,c)=1$. Examination of the spectrum of elementary string states shows that there is a unique half BPS supermultiplet associated with the charge $(Q,0)$ and hence also with the charges (aQ,cQ). Thus we have $d_h(aQ,cQ)=\delta_{Q^2,0}$, and in particular

$$d_h(Q,0)=\delta_{Q^2,0}, \qquad d_h(0,P)=\delta_{P^2,0}\,. \tag{4.5}$$

Substituting this into (4.4) we get a result in clear agreement with (4.1).

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